Journal of Mathematical Sciences: Advances and Applications
Volume 4, Number 2, 2010, Pages 303-314

CONVERGENCE OF THE PROJECTION TYPE ISHIKAWA ITERATION PROCESS WITH ERRORS FOR A FINITE FAMILY OF NONSELF I_i- ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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Abstract

In this paper, we consider the strong convergence of the projection type Ishikawa iteration process to a common fixed point of a finite family of nonself I_i- asymptotically quasi-nonexpansive mappings. Our results of this paper improve and extend the corresponding results of Temir and Gul [10], Temir [11], and Thianwan [12].

1. Introduction

Throughout this paper, let C be a nonempty subset of a real normed linear space X and denote the set of all fixed points of a mapping T by

2010 Mathematics Subject Classification: 47H10, 47H76.

Keywords and phrases: nonself I_i - asymptotically quasi-nonexpansive mapping, common fixed point, projection type Ishikawa iteration process.

This work was supported by the Scientific Research Foundation of Kunming University.

Received November 6, 2009

$F(T)$, the n-th iterate $T(T(\dots, Tx)\dots)$ of T by $T^n x$ and $T^0 = E$, where E denotes the mapping $E : C \to C$ defined by $Ex = x$, respectively.

Let T be a self-mapping of C. T is said to be *asymptotically nonexpansive*, if there exists a real sequence $\{\lambda_n\} \subset [0, +\infty)$, with $\lim_{n\to\infty} \lambda_n = 0$, such that $\|T^n x - T^n y\| \le (1 + \lambda_n)\|x - y\|$, $\forall x, y \in C$. A mapping T is called *uniformly L-Lipschitzian*, if there exists a real number $L > 0$ such that $\|T^n x - T^n y\| \le L\|x - y\|$, for every $x, y \in K$ and each $n \ge 1$.

It was proved in [2] that if X is uniformly convex, and if C is bounded closed and convex subset of X, then every asymptotically nonexpansive mapping has a fixed point.

T is called *I-asymptotically quasi-nonexpansive* on C, if there exists sequence $\{v'_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} v'_n = 0$ such that $\|T^n u - p\| \le (v'_n + 1)\|I^n u - p\|$, for all $u \in C$, $p \in F(T) \cap F(I)$, and $n = 1, 2, \cdots$.

Remark 1.1. From above definitions, it is easy to see, if $F(T)$ is nonempty, an asymptotically nonexpansive mapping must be I-asymptotically quasi-nonexpansive. It is obvious that, an asymptotically nonexpansive mapping is also uniformly L-Lipschitzian with $L = \sup\{1 + v_n : n \ge 1\}$. However, the converses of these claims are not true in general.

In the past few decades, many results on fixed points on asymptotically nonexpansive, quasi-nonexpansive, and asymptotically quasi-nonexpansive mappings in Banach space and metric spaces are obtained (see, e.g., [7, 9]). Recently, Rhoades and Temir [5] studied the convergence theorems for I-nonexpansive mappings, Temir and Gul [10] studied the convergence theorems for I-asymptotically quasi-nonexpansive mapping in Hilbert space. Very recently, Temir [11] studied the convergence theorems of implicit iteration process for a finite family of I-asymptotically nonexpansive mappings.

In most papers [1, 4, 9], which concern the iteration methods, the Ishikawa iteration scheme as follows: for any given $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= a_n S y_n + b_n x_n, \\ y_n &= a'_n T x_n + b'_n x_n, \end{aligned} \quad n \geq 1, \tag{1.1}$$

where $\{a_n\}, \{b_n\}, \{a'_n\}$, and $\{b'_n\}$ are real sequences in $[0, 1)$ with $a_n + b_n = 1 = a'_n + b'_n$ are bounded sequences in C.

On one hand, S, T have been assumed to map C into itself in (1.1), and the convexity of C ensures that the sequence $\{x_n\}$ given by (1.1) is well defined. If, however, C is a proper subset of the real Banach space X and T maps C into X, then the sequence given by (1.1) may not be well defined. One method that has been used to overcome this in the case of single mapping T is to generalize the iteration scheme by introducing a retraction $P : X \to C$ in the recursion formula (1.1). For nonself nonexpansvie mappings, some authors (see, e.g., [8, 14]) have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach spaces.

As an important generalization of the class of asymptotically nonexpansive self-mappings, Chidume [1] in 2003, generalized nonexpansive, asymptotically nonexpansive, uniformly L-Lipschitzian to

Definition 1.1. Let C be a nonempty subset of a real normed space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself mapping $T : C \to X$ is called *asymptotically*, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \in \mathcal{N}$,

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n \|x - y\|, \quad \text{for every } x, y \in C.$$

T is said to be *uniformly L-Lipschitzian*, if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|, \quad \text{for every } x, y \in C.$$

And if let $T, I : C \to X$, the mapping T is said to be Γ-*Lipschitzian*, if there exists $\Gamma \geq 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq \Gamma\|I(PI)^{n-1}x - I(PI)^{n-1}y\|, \quad \text{for every } x, y \in C.$$

In 2006, Wang [13] generalized the work to prove strong and weak convergence theorems for a pair of nonself asymptotically nonexpansive mappings.

On the other hand, in 1991, Schu [7] introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space as follow:

$$x_{n+1} = (1 - a_n)x_n + a_n T^n x_n. \tag{1.2}$$

Since then, Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space or Banach spaces [4, 6].

In 2009, Thianwan [12] generalized their work to prove strong and weak convergence theorems of projection type Ishikawa iteration for a pair of nonself asymptotically nonexpansive mappings.

Motivated by above works, in this paper, we consider the following projection type Ishikawa iteration process with errors (1.3) to approximating common fixed points for a finite family of nonself I_i- asymptotically quasi-nonexpansive mappings T_i, and obtain the strong convergence theorems for such mappings in uniformly convex Banach spaces.

Definition 1.2. Let $T_i : X \to C$, $i \in \{1, \ldots, N\}$, T_i is nonself I_i-asymptotically quasi-nonexpansive mappings, I_i is nonself asymptotically nonexpansive. Then, an iterative scheme is the sequences of mappings $\{x_n\}$ defined by, for given $x_1 \in C$,

$$\begin{aligned} x_{n+1} &= P(a_n I_{i(n)}(PI_{i(n)})^{k(n)-1} y_n + b_n y_n + c_n u_n), \\ y_n &= P(a'_n T_{i(n)}(PT_{i(n)})^{k(n)-1} x_n + b'_n x_n + c'_n v_n), \end{aligned} \quad n \geq 1, \tag{1.3}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$, and $\{c'_n\}$ are real sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, $n = (k(n) - 1)N + i(n)$, $i(n) \in \{1, \ldots, N\}$, and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

We restate the following definitions and lemmas, which play an important roles in our proofs.

Definition 1.3. Let X be a Banach space, C be a nonempty subset of X. Let $T : C \to C$. Then T is said to be

(1) *demiclosed* at y, if whenever $\{x_n\} \subset C$ such that $x_n \rightharpoonup x \in C$ and $Tx_n \to y$, then $Tx = y$.

(2) *semi-compact*, if for any bounded sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some x^* in K.

(3) *completely continuous*, if the sequence $\{x_n\}$ in C converges weakly to x_0 implies that $\{Tx_n\}$ converges strongly to Tx_0.

Lemma 1.1 [2]. *Let* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, *and* $\{\mu_n\}$ *be four nonnegative real sequences satisfying* $\alpha_{n+1} \leq (1 + \gamma_n)(1 + \mu_n)\alpha_n + \beta_n$, *for all* $n \geq 1$. *If* $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, *and* $\sum_{n=1}^{\infty} \beta_n < \infty$, *then* $\lim_{n\to\infty} \alpha_n$ *exists.*

Lemma 1.2 [7]. *Let E be a real uniformly convex Banach space and* $0 \leq p \leq t_n \leq q < 1$, *for all positive integer* $n \geq 1$. *Also, suppose* $\{x_n\}$ *and* $\{y_n\}$ *are two sequences of E such that* $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$, *and* $\limsup_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$ *hold for some* $r \geq 0$, *then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.3 [1]. *Let X be a real uniformly convex Banach space, C be a nonempty closed subset of X, and let* $T : C \to X$ *be nonself asymptotically nonexpansive mapping with a sequence* $\{k_n\} \subset [1, \infty)$ *and* $k_n \to 1$ *as* $n \to \infty$. *Then,* $E - T$ *is demiclosed at zero.*

2. Main Results

Lemma 2.1. *Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of* X, $\{T_i : i \in \{1, 2, \ldots, N\}\} : K \to X$ *be* N *uniformly* Γ*-Lipschitzian* I_i*-asymptotically quasi-nonexpansive non-*

self-mappings with sequences $\{v_{in}\} \subset [0, \infty)$ *such that* $\sum_{n=1}^{\infty} v_{in} < \infty$, *and* $I_i : i \in \{1, \ldots, N\} : C \to X$ *be N uniformly L-Lipschitzian asymptotically nonexpansive nonself-mappings with* $\{u_{in}\} \subset [0, \infty)$ *such that* $\sum_{n=1}^{\infty} u_{in} < \infty$ *and* $F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. *Suppose that for any given* $x \in K$, *the sequence* $\{x_n\}$ *is generated by* (1.3), *where* $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$. *If* $F \neq \emptyset$, *then* $\lim_{n\to\infty} \|I_l x_n - x_n\| = \lim_{n\to\infty} \|T_l x_n - x_n\| = 0$, $\forall l = 1, 2, \ldots, N$.

Proof. Since C is bounded, there exists $M > 0$ such that $\|y_n - u_n\| \leq M$ and $M' > 0$ such that $\|x_n - v_n\| \leq M'$ for all $n \in \mathcal{N}$. For any $p \in F = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$.

$$\begin{aligned}
\|x_{n+1} - p\| &= \|a_n I_{i(n)} (PI_{i(n)})^{k(n)-1} y_n + b_n y_n + c_n u_n - p\| \\
&\leq a_n \|I_{i(n)} (PI_{i(n)})^{k(n)-1} y_n - p\| + (1 - a_n) \|y_n - p\| + c_n \|u_n - y_n\| \\
&\leq (1 + a_n v_{ik}) \|y_n - p\| + c_n M. \qquad (2.1)
\end{aligned}$$

$$\begin{aligned}
\|y_n - p\| &= \|a'_n T_{i(n)} (PT_{i(n)})^{k(n)-1} x_n + b'_n x_n + c'_n v_n - p\| \\
&\leq a'_n \|T_{i(n)} (PT_{i(n)})^{k(n)-1} x_n - p\| + (1 - a'_n) \|x_n - p\| + c'_n \|v_n - x_n\| \\
&\leq (1 - a'_n) \|x_n - p\| + a'_n (1 + u_{ik})(1 + v_{ik}) \|x_n - p\| + c'_n M' \\
&\leq [1 + a'_n (u_{ik} + v_{ik} + u_{ik} v_{ik})] \|x_n - p\| + c'_n M'. \qquad (2.2)
\end{aligned}$$

Transposing and simplifying above inequality and noticing that $a_n \in [\delta, 1 - \delta]$. We have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 + a_n v_{ik}) [1 + a'_n (u_{ik} + v_{ik} + u_{ik} v_{ik}) \|x_n - p\| + c'_n M'] + c_n M \\
&\leq (1 + \gamma_n)(1 + \mu_n) \|x_n - p\| + \beta_n, \qquad (2.3)
\end{aligned}$$

where $\gamma_n = a_n v_{ik}$, $\mu_n = a'_n (u_{ik} + v_{ik} + u_{ik} v_{ik})$, $\beta_n = c'_n M' + c_n M$.

Since, $\sum_{k=1}^{\infty} u_{ik} < \infty$, $\sum_{k=1}^{\infty} v_{ik} < \infty$ for all $i \in \{1, 2, \ldots, N\}$, and $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$ for all $n \in \mathcal{N}$, thus, $\sum_{k=1}^{\infty} \gamma_n < \infty$, $\sum_{k=1}^{\infty} \mu_n < \infty$, and $\sum_{k=1}^{\infty} \beta_n < \infty$.

By Lemma 1.1, $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in F$. Let $\lim_{n\to\infty}\|x_{n+1} - p\| = d > 0$. Since, $\|I_{i(n)}(PI_{i(n)})^{k(n)-1} y_n - p + c_n(u_n - y_n)\| \leq (1 + v_{ik})\|y_n - p\| + c_n M$. By (2.2), we have $\limsup_{n\to\infty}\|I_{i(n)} (PI_{i(n)})^{k(n)-1} y_n - p + c_n(u_n - y_n)\| \leq d$. And $\|y_n - p + c_n(u_n - y_n)\| \leq \|y_n - p\| + c_n M$, which implies $\limsup_{n\to\infty}\|y_n - p + c_n(u_n - y_n)\| \leq d$.

$\lim_{n\to\infty}\|x_{n+1} - p\| = d$ means that $\lim_{n\to\infty}\|a_n[I_{i(n)}(PI_{i(n)})^{k(n)-1} y_n - p + c_n(u_n - y_n)] + (1 - a_n)[y_n - p + c_n(u_n - y_n)]\| = d$. By Lemma 1.2, we have

$$\lim_{n\to\infty}\|I_{i(n)}(PI_{i(n)})^{k(n)-1} y_n - y_n\| = 0. \tag{2.4}$$

Using (2.1), $\|x_{n+1} - p\| \leq (1 + a_n v_{ik})\|y_n - p\| + c_n M$. We have $d = \lim_{n\to\infty}\|x_{n+1} - p\| \leq \liminf_{n\to\infty}\|y_n - p\|$. It follows from (2.2), $\limsup_{n\to\infty}\|y_n - p\| \leq \lim_{n\to\infty}\|x_{n+1} - p\| = d$ that $\lim_{n\to\infty}\|y_n - p\| = d$.

This implies that

$$\lim_{n\to\infty}\|y_n - p\| = \lim_{n\to\infty}\|a'_n(T_{i(n)}(PT_{i(n)})^{k(n)-1} x_n - p + c'_n(v_n - x_n)) + (1 - a'_n)(x_n - p + c'_n(v_n - x_n))\| = d.$$

Since, $\|T_{i(n)}(PT_{i(n)})^{k(n)-1} x_n - p + c'_n(v_n - x_n)\| \leq (1 + u_{ik})(1 + v_{ik})\|x_n - p\| + c'_n M'$ and $\|x_n - p + c'_n(v_n - x_n)\| \leq \|(x_n - p)\| + c'_n M'$, we have $\limsup_{n\to\infty}\|T_{i(n)}(PT_{i(n)})^{k(n)-1} x_n - p + c'_n(v_n - p)\| \leq d$ and $\limsup_{n\to\infty} \|x_n - p + c'_n(v_n - p)\| \leq d$. By Lemma 2.2, we have

$$\lim_{n\to\infty}\|T_{i(n)}(PT_{i(n)})^{k(n)-1} x_n - x_n\| = 0. \tag{2.5}$$

$\|y_n - x_n\| = \|a'_n T_{i(n)}(PT_{i(n)})^{k(n)-1} x_n + b'_n x_n + c'_n v_n - x_n\|$

$$\leq a'_n \|T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n - x_n\| + c'_n\|v_n - p\| \to 0 \text{ (as } n \to \infty). \tag{2.6}$$

Also,

$$\begin{aligned} &\|I_{i(n)}(PI_{i(n)})^{k(n)-1}x_n - x_n\| \\ &\quad \leq \|I_{i(n)}(PI_{i(n)})^{k(n)-1}x_n - I_{i(n)}(PI_{i(n)})^{k(n)-1}y_n\| \\ &\quad + \|I_{i(n)}(PI_{i(n)})^{k(n)-1}y_n - y_n\| + \|y_n - x_n\| \\ &\quad \leq (1 + L)\|y_n - x_n\| + \|I_{i(n)}(PI_{i(n)})^{k(n)-1}y_n - y_n\|. \end{aligned}$$

Thus, it follows from (2.4) and (2.6), that

$$\lim_{n\to\infty}\|I_{i(n)}(PI_{i(n)})^{k(n)-1}x_n - x_n\| = 0. \tag{2.7}$$

In addition,

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|a_n I_{i(n)}(PI_{i(n)})^{k(n)-1}y_n + b_n y_n + c_n u_n - x_n\| \\ &\leq a_n\|I_{i(n)}(PI_{i(n)})^{k(n)-1}y_n - y_n + y_n - x_n\| + c_n\|u_n - y_n\| \\ &\qquad + (b_n + c_n)\|y_n - x_n\| \\ &\leq a_n\|I_{i(n)}(PI_{i(n)})^{k(n)-1}y_n - y_n\| + c_n M + \|y_n - x_n\|, \end{aligned}$$

by (2.4) and (2.6), we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, as well as for all $l \in \{1, 2, \dots, N\}$

$$\lim_{n\to\infty}\|x_n - x_{n+l}\| = 0. \tag{2.8}$$

Notice that for each $n > N$, $n = (n - N)(\text{mod } N)$, and $n = (k(n) - 1)N + i(n)$, hence $n - N = ((k(n) - 1) - 1)N + i(n) = (k(n - N) - 1)N + i(n - N)$, that is, $k(n - N) = k(n) - 1$ and $i(n - N) = i(n)$.

From (2.5), (2.7), and (2.8),

$$\|x_n - T_n x_n\| \leq \|x_n - T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\| + \|T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n - T_n x_n\|$$

$$\leq \|x_n - T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\| + \Gamma\|I_{i(n)}(PI_{i(n)})^{k(n)-1}x_n$$

$$+ I_{i(n-N)}(PI_{i(n-N)})^{k(n)-1}x_{n-N}$$

$$- I_{i(n-N)}(PI_{i(n-N)})^{k(n)-1}x_{n-N} - x_{n-N} + x_{n-N} - x_n\|$$

$$\leq \|x_n - T_{i(n)}(PT_{i(n)})^{k(n)-1}x_n\|$$

$$+ \Gamma\|I_{i(n-N)}(PI_{i(n-N)})^{k(n)-1}x_{n-N} - x_{n-N}\|$$

$$+\Gamma(1 + L)\|x_n - x_{n-N}\| \to 0 \ (n \to \infty).$$

This implies that $\lim_{n\to\infty}\|T_n x_n - x_n\| = 0$. Now, for all $l = \{1, 2, \dots, N\}$.

$$\|x_n - T_{n+l}x_n\| \leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\|$$

$$\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \Gamma L\|x_{n+l} - x_n\|$$

$$\leq \|x_{n+l} - T_{n+l}x_{n+l}\| + (1 + \Gamma L)\|x_n - x_{n+l}\| \to 0 \ (n \to \infty).$$

So, $\lim_{n\to\infty}\|T_{n+l}x_n - x_n\| = 0$, for all $l = \{1, 2, \dots, N\}$.

Consequently, we have

$$\lim_{n\to\infty}\|T_l x_n - x_n\| = 0. \tag{2.9}$$

$$\|x_n - I_n x_n\| \leq \|x_n - I_{i(n)}(PI_{i(n)})^{k(n)-1}x_n\| + \|I_{i(n)}(PI_{i(n)})^{k(n)-1}x_n - I_n x_n\|$$

$$\leq \|x_n - I_{i(n)}(PI_{i(n)})^{k(n)-1}x_n\| + L\|I_{i(n)}(PI_{i(n)})^{k(n)-2}x_n$$

$$- I_{i(n-N)}(PI_{i(n-N)})^{k(n)-2}x_{n-N} + I_{i(n-N)}(PI_{i(n-N)})^{k(n)-2}x_{n-N}$$

$$-x_{n-N} + x_{n-N} - x_n\|$$

$$\leq \|x_n - I_{i(n)}(PI_{i(n)})^{k(n)-1}x_n\|$$

$$+ L\|I_{i(n-N)}(PI_{i(n-N)})^{k(n)-2}x_{n-N} - x_{n-N}\|$$

$$+L(1 + L)\|x_n - x_{n-N}\| \to 0 \ (n \to \infty).$$

This implies that

$$\lim_{n\to\infty}\|I_n x_n - x_n\| = 0. \tag{2.10}$$

And

$$\|x_n - I_{n+l}x_n\| \le \|x_n - x_{n+l}\| + \|x_{n+l} - I_{n+l}x_{n+l}\| + \|I_{n+l}x_{n+l} - I_{n+l}x_n\|$$
$$\le (1 + L)\|x_n - x_{n+l}\| + \|x_{n+l} - I_{n+l}x_{n+l}\|.$$

Taking $\lim_{n\to\infty}$ on both sides in the above inequality, then we get $\lim_{n\to\infty}\|x_n - I_{n+l}x_n\| = 0,$ for all $l \in \{1, 2, \ldots, N\}$.

Consequently, we have

$$\lim_{n\to\infty}\|I_l x_n - x_n\| = 0. \tag{2.11}$$

The proof is completed.

Theorem 2.2. *Let X be a uniformly convex Banach space and $C, T_i, I_i, \{x_n\}$ be same as in Lemma* 2.1. *If one of I_i, let I_m is a semi-compact mapping and $F \ne \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$ and $\{I_i\}$.*

Proof. Since, I_m is semi-compact mapping, $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - I_m x_n\| = 0,$ then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges to x^*. It follows from Lemma 1.3, $x^* \in F(I_m)$. In addition, since T_i is a uniformly Γ-Lipschitzian mapping and I_i is a uniformly L-Lipschitzian mapping, $\lim_{n\to\infty}\|x_n - I_i x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$. So, $\|x^* - I_i x^*\| = 0,$ and $\|x^* - T_i x^*\| = 0$. This implies that $x^* \in F(I_i) \cap F(T_i)$. Since, the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to x^* and $\lim_{n\to\infty}\|x_n - x^*\|$ exists, then $\{x_n\}$ converges strongly to the common fixed point $x^* \in F$. The proof is completed.

Theorem 2.3. *Let X be a uniformly convex Banach space and C, T_i, I_i, $\{x_n\}$ be same as in Lemma 2.1. If one of I_i, let I_m is completely continuous mapping and $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$ and $\{I_i\}$.*

Proof. By Lemma 2.1, $\{x_n\}$ is bounded. Since, $\lim_{n\to\infty}\|x_n - I_l x_n\| = \lim_{n\to\infty}\|x_n - T_l x_n\| = 0$, then $\{T_l x_n\}$ and $\{I_l x_n\}$ are bounded. Since I_m is completely continuous, that exists subsequence $\{I_m x_{n_j}\}$ of $\{I_m x_n\}$ such that $\{I_m x_{n_j}\} \to p$ as $j \to \infty$. Thus, we have $\lim_{j\to\infty}\|x_{n_j} - I_m x_{n_j}\| = 0$. Hence, by the continuity of I_m and Lemma 1.3, we have $\lim_{j\to\infty}\|x_{n_j} - p\| = 0$ and $p \in F(I_m)$. Further, for all $i \in \{1, 2, \ldots, N\}$

$$\|I_i x_{n_j} - p\| \leq \|I_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - I_m x_{n_j}\| + \|I_m x_{n_j} - p\|,$$

$$\|T_i x_{n_j} - p\| \leq \|T_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - I_m x_{n_j}\| + \|I_m x_{n_j} - p\|.$$

Thus, $\lim_{n_j\to\infty}\|I_i x_{n_j} - p\| = 0$ and $\lim_{n_j\to\infty}\|T_i x_{n_j} - p\| = 0$. This implies that $\{I_i x_{n_j}\}$, $\{T_i x_{n_j}\}$ converges strongly to p. Since, I_i is a uniformly L-Lipschitzian mapping, T_i is uniformly Γ-Lipschitzian, so I_i, T_i is continuous. So, $p = I_i p = T_i p$. Hence, $p \in F$. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - p\|$ exists. Thus, $\lim_{n\to\infty}\|x_n - p\| = 0$. The proof is completed.

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